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Solvable lattice models and character formulas

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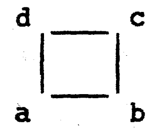
このノートでは二次元正方格子上の可解な統計モデルの構成法、及びそのモデルの local height probability (LHP) の計算結果についてのべる。

Baxterにより見い出された 8 vertex SOS モデルと呼ばれる可解なモデルに、"fusion" と呼ぶ手続きを施すことにより、新しい可解なモデルが構成される。又、このモデルのLHPはアフィン・リー環 $A_1^{(1)}$ のテンソル積表現の指標の既約分解の係数となることが示される。

In this note we consider a statistical model on a two dimensional lattice, especially solvable models. Our purpose is to calculate the quantity called local height probability (LHP). and explain its automorphic property by the representation theory of affine Lie algebras.

1. Let $\mathcal{L}$ be a two dimensional square lattice of finite size. Fix a positive integer $L \geq 4$. To each site i,(that is a lattice point of $\mathcal{L}$), we attach a variable ℓ_i taking integer values between 1 and L-1. Namely we consider the (L-1)-state model. We call this ℓ_i the height variable or simply height. For each height configuration round a face 

, we assign a Boltzmann weight $W(a,b,c,d)$. Hereafter we mean by face a square formed by four neighboring sites. A Boltzmann weight is an unnormalized probability of a height configuration around a face. In this way with heights and weights we obtain a statistical model on $\mathcal{L}$, sometimes called interaction-round-a-face (IRF) model[1].

One of the fundamental problems in the statistical mechanics is to derive macroscopic quantities (such as the spontaneous magnetization) from the microscopic ones (Boltzmann weights and so on) and study the critical behaviours of the model. Mathematically this amount to the calculation of the quantities such as the free energy, LHPs and (if possible) multi-point correlation functions in the thermodinamic limit $|\mathcal{L}| \longrightarrow \infty$, where $|\mathcal{L}|$ denote the size of the lattice (= the number of sites of $\mathcal{L}$). These quantities are defined as follows:

a) free energy

$$f = \lim_{|\mathcal{L}|\to\infty} \frac{1}{|\mathcal{L}|} \log Z,$$

where Z is the partition function for a finite lattice

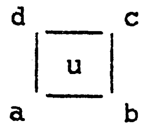
$$Z = \sum_{\{\text{config.}\}} \prod_{\{\text{face}\}} W(\ell_i, \ell_j, \ell_m, \ell_n).$$

b) LHP

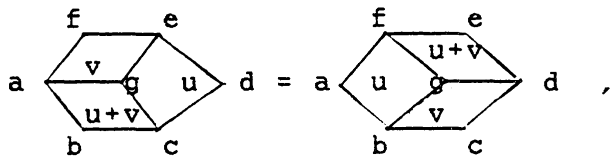
$$P_a = \lim_{|\mathcal{L}|\to\infty} Z^{-1} \sum_{\{\text{config.}\}} \delta(\ell_1, a) \prod_{\{\text{face}\}} W(\ell_i, \ell_j, \ell_m, \ell_n).$$

Here the sum is over all allowed arrangements of heights on $\mathcal{L}$, the product is over all faces, and ℓ_1 denotes the center height. In other words, the LHP is the probability of the center height being equal to a. The critical behaviours are extracted from the singular behaviours of these quantities.

In general these quantities are very difficult to calculate. A model is called solvable or integrable when one can compute free energy and, if possible, the LHP and correlation functions. We are interested in the structure of solvable models. First of all, we must find solvable models. There is a sufficient condition for solvability. Namely, if the Boltzmann weights of a model satisfy the so called star-triangle relation (STR), there is a standard method to calculate the free energy and the LHP (the method of the commuting transfer matrix and the corner transfer matrix)[1]. The STR is a system of algebraic equations for Boltzmann weights, which we explain next. Our first result is the construction of a solution (=weights) to the STR.

2. The STR is expressed graphically as follows. Here we regard weights depend on extra parameter u (which we call the spectral parameter) and denote as 

$= W(a,b,c,d)$. Then the equation is



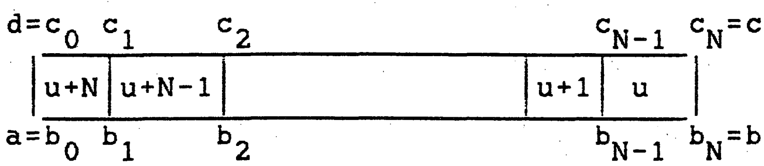
for any a,b,c,d,e,f and $u,v \in \mathbb{C}$. That is, take the product of 3 weights and sum over center height g in both hand sides and require them to be equal. This is an overdetermined system: the number of equations is $(L-1)^6$ and the number of unknowns is $(L-1)^4$.

Our strategy for constructing a solution to the STR is as follows[2]. We start with the known solution to the STR presented by Baxter in 1973 [3], which is called the eight vertex SOS model (8VSOS). In this model heights take integer values with the restriction $\ell_i - \ell_j = \pm 1$ for adjacent sites i,j. These weight are expressed in terms of elliptic theta function

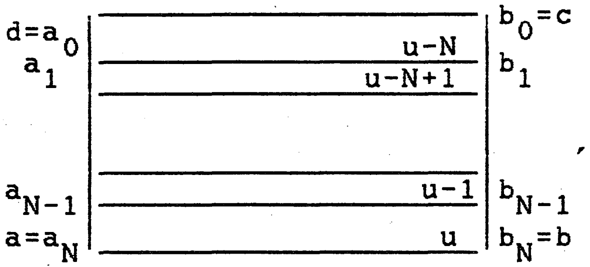
$$\Theta(u) = 2p^{1/4}\sin(\pi u/2) \prod_{n=1}^{\infty} (1-2p^n\cos \pi u+p^{2n})(1-p^{2n})^2 ,$$

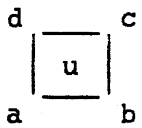
and contain other two parameters λ and ξ.

Using these weights of 8VSOS, we construct new weights that also satisfy the STR. We call this process "fusion". For that purpose, we introduce a positive integer N. The new weights satisfy the condition i) $\ell_i - \ell_j = -N, -N+2, \cdots, N-2, N$ for adjacent sites. Roughly, weights are constructed as follows. For detail, we refer to [2]. Prepare N weights of 8VSOS with parameters marked in the box and sum over a_i's and divide out common zeros.



Here a_i's satisfy the condition $|a_i - a_{i+1}| = 1$. The STR implies that the result is independent of the choice of c_i's provided that $|c_i - c_{i+1}| = 1$. Further summing over these rectangles with respect to a_i's



we obtain the weight 

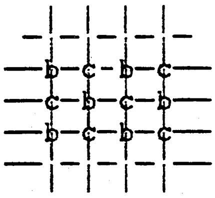
. Again the STR for the 8VSOS implies that these new weights satisfy the STR. The resulting weights are N-th order polynomials in $\theta(u)$. By multiplying a suitable factor, the resulting weights have the following symmetries

$$\begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array} = \begin{array}{ccc} b & & a \\ & \boxed{u} & \\ c & & d \end{array} = \frac{g_a\ g_c}{g_b\ g_d} \begin{array}{ccc} c & & d \\ & \boxed{-1-u} & \\ b & & a \end{array} ,$$

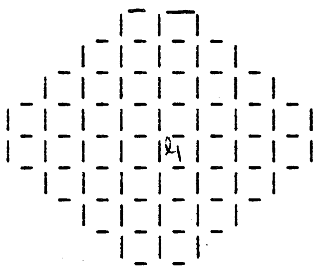
where $g_a = \varepsilon_a \sqrt{\theta(\lambda(\xi+a))}$, $\varepsilon_a^{\ 2} = 1$, $\varepsilon_a\ \varepsilon_{a+1} = (-)^a$, and λ and ξ are the parameters in the weights of the 8VSOS. However there remains infinite sorts of weights. To remedy this situation we introduce another positive integer L such that $L \geq N + 3$ and impose another condition on adjacent heights: ii) $N < \ell_i + \ell_j < 2L - N$. This is achieved by specializing λ and ξ. In this way only a finite sorts of weights are singled out and they are again shown to satisfy the STR among themselves. We note that the conditions i) and ii) imply $\ell_i \in \{1,\cdots,L-1\}$ and also that the condition ii) is necessary to exclude weights which have poles after the specialization of λ and ξ. Thus we obtain a $(L-1)$-state solvable model. For $N = 1$ such a restriction with the introduction of L was considered by Andrews-Baxter-Forrester (ABF). They called the resulting model restricted eight vertex SOS model.

3. From now on we consider the parameter region $0 < p < 1$, $-1 < u < 0$ (regime III in the terminology of ABF) and we calculate the LHP of our model.

For the calculation of the LHP we need to know "ground states". A ground state is, by definition, an arrangement of heights that maximizes the summand in the definition of the partition function. For our model, a ground state consists of an arrangement of 2 values of heights alternating from site to site.

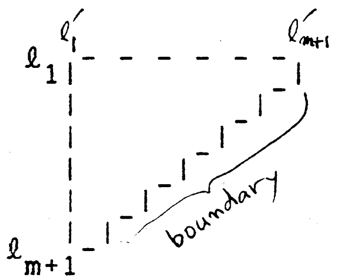


In the calculation of the LHP we assume that values of boundary heights take those of a ground states, say b,c. Then Baxter's corner transfer matrix method supplies the following expression for the LHP in the limit $|\mathcal{L}| \longrightarrow \infty$. The method of the CTM proceed as follows. Place the lattice like this



and let us denote by A the matrix of partition functions for this quadrant with prescribed heights on the left and the upper lines and on the boundary. The values of heights on the left and the upper lines labels elements of the matrix.

$A_{\ell,\ell'} \longleftrightarrow$



$$\ell = (\ell_1, \cdots, \ell_{m+1}),$$
$$\ell' = (\ell'_1, \cdots, \ell'_{m+1}).$$

We define B, C, D similarly. Then the partition function and the LHP are expressed as traces of matrices:

$$Z = \mathrm{tr}(ABCD), \qquad P(a|b,c) = \lim_{|\mathcal{L}|\to\infty} \mathrm{tr}_{\ell_1=a}(ABCD)/\mathrm{tr}(ABCD).$$

The STR tells us that A takes the simple form $A = a(u)e^{u\mathcal{H}}$ in the limit $|\mathcal{L}| \longrightarrow \infty$, where the matrix $\mathcal{H}$ is indepent of u and $a(u)$ is a scalar function. The weights have periodicity with respect to u, for they are expressed in terms of theta functions. This means that eigenvalues of A are integer multiples of some fixed constant. Taking the symmetry of weights into account, we have

$$P(a|b,c) = x^{-\lambda_{a,L}} G_{a,L}(x)\ X(a|b,c)/M_{b,c},$$

$$M_{b,c} = \sum_{a=1}^{L-1} x^{-\lambda_{a,L}} G_{a,L}(x)\ X(a|b,c).$$

Here $\lambda_{a,L} = (2a-L)^2/8L$, $G_{a,L}(x) = \sum_{\nu\in\mathbb{Z}} (-)^{\nu}\, x^{(2a-L+2L\nu)^2/8L}$ and $X(a|b,c)$ is the one dimensional partition sum

$$X(a|b,c) = \lim_{m\to\infty} X_m(a|b,c),$$

$$X_m(a|b,c) = \sum_{\{\ell_2,\cdots,\ell_m\}} x^{\varphi_m(\ell_1,\cdots,\ell_{m+2})},$$

$$\varphi_m(\ell_1,\cdots,\ell_{m+2}) = \sum_{j=1}^{m} j|\ell_j - \ell_{j+2}|/2,$$

where $\ell_1 = a$ and $\ell_{m+1} = b,c$, $\ell_{m+2} = c,b$ for m = even, odd, respectively. The sum is over ℓ_i' s that satisfy $|\ell_i - \ell_{i+1}| = 1$. The parameter x relates to the nome p through the relation $p = e^{-\varepsilon}$, $x = e^{-4\pi^2/\varepsilon L}$. From this we easily derive the recurrence relation for X_m and consequently a series expansion of X. However, this series expansion is not useful for the study of critical behaviours, which correspond to $p \longrightarrow 0$. As has been shown by examples[1] the LHPs of solvable models exhibit modular properties that relates the behaviours near $x \longrightarrow 1$ and $p \longrightarrow 0$. Our second result is that our model also have this property and that this automorphic property comes from that of characters of affine Lie algebras[4].

4. To state our result, we need some notations. The affine Lie algebra $A_1^{(1)}$ is isomorphic to a central extension of the Lie algebra $sl(2,\mathbb{C}[t,t^{-1}])$ over Laurent polynomials with the bracket $[a(m),b(n)] = [a,b](m+n) + m\,\delta_{m+n,0}\,\mathrm{tr}(ab)\,c$. Here we set $a(m) = a \otimes t^m$ for $a \in sl(2,\mathbb{C})$ and c is the central element. This $A_1^{(1)}$ has the generators

$$h_0 = c - h_1,\ h_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ e_0 = \begin{pmatrix} & \\ t & \end{pmatrix},\ e_1 = \begin{pmatrix} & 1 \\ & \end{pmatrix},$$
$$f_0 = \begin{pmatrix} & t^{-1} \\ & \end{pmatrix},\ f_1 = \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

Let $V_{k,m}$ $(0 \le k \le m)$ be the irreducible highest weight $A_1^{(1)}$ module of level m and spin $k/2$: that is the module generated from the highest weight vector v such that $e_i v = 0$, $i = 0,1$, $cv = mv$, $h_1 v = kv$. We consider a pair of affine Lie algebras $A_1^{(1)} + A_1^{(1)} \supset \Delta(A_1^{(1)})$. Here Δ signifies the diagonal embedding. Then the irreducible decomposition of tensor modules with respect to the diagonal action of $A_1^{(1)}$ takes the following form

$$V_{k_1,m_1} \otimes V_{k_2,m_2} = \bigoplus_{k_3=0}^{m_3} W_{k_3}, \quad m_3 = m_1 + m_2.$$

Here W_k is isomorphic to a direct sum of copies of V_{k_3,m_3} and $V_{k_3,m_3} = \{ 0 \}$ if $k_3 \equiv k_1 + k_2 \pmod{2}$.

We set $\Omega_k = \{ v \in W_k \mid \Delta(e_i)v = 0,\ \Delta(h_1)v = kv \}$. This is the space of highest weight vectors in W_k. We have $W_{k_3} = \Omega_{k_3} \otimes V_{k_3,m_3}$. Let $\chi_{k,m}(z,q)$ be the character of $V_{k,m}$ (it differs from the usual one by a fractional power of q). Explicitly it is given as a quotient of theta functions

$$\chi_{k,m}(z,q) = F_{k+1,m+2}(z,q)/F_{1,2}(z,q),$$

$$F_{j\ell}(z,q) = \sum_{\gamma\in\mathbb{Z}+j/2\ell} q^{\ell\gamma^2}(z^{-\ell\gamma} - z^{\ell\gamma}).$$

Here $z = e^{-\alpha_1}$, $q = e^{-\alpha_0-\alpha_1}$ and α_i are simple roots of $A_1^{(1)}$. Further let $B_{k_1,k_2;k_3}(q)$ be the character of Ω_{k_3} (this is independent of z, since on Ω_{k_3} the value of h_1 is constant. For the precise definition, we refer to [4]). In terms of characters the above decomposition reads as

$$\chi_{k_1,m_1}(z,q)\ \chi_{k_2,m_2}(z,q) = \sum_{k_3=0}^{m_3} B_{k_1,k_2;k_3}(q)\ \chi_{k_3,m_3}(z,q).$$

Then our result is expressed as

$$P(a|b,c) = \frac{G_{k_3,m_3}(x)\ G_{1,2}(x)}{G_{k_1,m_1}(x)\ G_{k_2,m_2}(x)}\ B_{k_1,k_2;k_3}(x^2),$$

under the identificatin

$$k_1 = (b-c+N)/2,\ k_2 = (b+c-N)/2 - 1, k_3 = a - 1,$$

$m_1 = N,\ m_2 = L - N - 2,\ m_3 = L-2.$

Here $G_{j,\ell}(x) = F_{j,\ell}(x,x^2)$. The conditions i), ii) on heights means precisely $0 \le k_i \le m_i$. We note that the above identity characterizes the B's uniquely: Theta functions with fixed order form a linear space. Since the product of theta functions on the left hand side is again a theta function, it is expressed as a linear combination of elements of a basis of the space of theta functions of given order. The B's are the coefficients of this linear combination. The LHPs in other regimes are also characterized by theta function identities[5]. By the result of Kac-Peterson [6], the characters $\chi_{k,m}$ have automorphic properties which originate from those of theta functions. This implies the automorphic properties of B's. Using these properties, we can easily read off critical behaviours of our model. The critical exponents of our model include those of known integrable conformal field theories (for detail we refer to [4]). The role of $A_1^{(1)}$ in our lattice model is yet to be clarified. That must be an interesting problem by itself as well as its relation to integrable conformal field theories.

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